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GEOMETRIC INTERPRETATION OF DISCRETE FRACTIONAL ORDER CONTROLLERS BASED ON SAMPLING TIME SCALING PROPERTY AND EXPERIMENTAL VERIFICATION OF FRACTIONAL $1/S^{\alpha}$ SYSTEMS' ROBUSTNESS

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ABSTRACT

This article proposes a geometric interpretation of discrete fractional order controllers based on sampling time scaling property. Due to its clear interpretation, satisfactory accuracy and easy programming, the property could be used as a reliable simulation and realization method for fractional order control systems. The experiments of one-inertia speed control with fractional order integral controllers realized by the proposed sampling time property are also carried out to verify the theoretical robustness of fractional $1/s^{\alpha}$ *systems. The experimental results show the superior robustness performances of fractional* $1/s^{\alpha}$ *systems against saturation non-linearity and inertia variation, which highlights the promising aspects of fractional order control.*

1 INTRODUCTION

The concept of expanding derivatives and integrals to fractional (non-integer) orders is by no means new. In fact, *Leibniz* mentioned this concept in a letter to $L'Hospital$ over three hundred years ago (1695) and the earliest more or less systematic studies seem to have been made in the beginning and middle of the 19th century by *Liouville*(1832), *Holmgren*(1864) and *Riemann*(1953) [1]. However, the concept of fractional order control, in which the controlled systems or controllers are described by fractional order differential equations, was not widely incorporated into control engineering mainly due to the conceptually difficult idea of taking fractional order and to the existence of so few physical applications at that time [2].

In last few decades, researchers pointed out that fractional order differential equations could model various real materials more adequately than integer order ones and provide an excellent tool for the description of dynamical processes [1, 3, 4]. Those fractional order models need the corresponding fractional order controllers to be proposed and evoked the interest to various applications of fractional order control [5–8]. The significance of fractional order control is that it is a generalization and "interpolation" of classical integral order control theory, which could lead to more adequate modeling and easier design of robust control systems against uncertainties.

It is well known that integer order derivatives and integrals have clear physical and geometric interpretations, such as slope or velocity for derivatives and area or distance for integrals generally. These clear and easily understandable interpretations simplified their applications to various problems in different fields, including control theory that is extremely well developed based on integer order differential equations. On the contrary, for fractional order derivatives and integrals, it was not so. The notorious lack of clear geometric interpretations made fractional order derivatives and integrals conceptually difficult and greatly obstructed their real applications. *Podlubny* proposed a simple geometric interpretation of fractional integrals as "changing shadows on the wall" and some pictures describing this changing process were given [9]. But for the applications of fractional order controllers to discrete control systems, how to interpret the role of these controllers in discrete domain is much more concerned.

The paper is organized as follows: in section II, basic mathematical aspects are mentioned in order to show that the fractional order control is a generalization of classical integer order control theory; in section III, an interpretation for discrete fractional order controllers based on "sampling time scaling" property is proposed and its application to reliable simulation of fractional order control systems is mentioned; in section IV, most emphasis is placed on the reliable realization of fractional order $1/s^{\alpha}$ one-inertia speed control system by proposed "sampling time scaling" property and its robustness against parameter variation and saturation non-linearity; finally, in section V, conclusions are drawn.

2 MATHEMATICAL ASPECTS

2.1 Mathematical Definitions

The mathematical definition of fractional calculus has been the subject of several different approaches [1,3]. The most frequently encountered definition is called *Riemann − Liouville* definition, in which the fractional order integrals are defined as

$$_aD_t^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\xi)^{\alpha-1} f(\xi)d(\xi) \tag{1}$$

while the definition of fractional order derivatives is

$$_aD_t^{\alpha} = \frac{d^{\gamma}}{dt^{\gamma}}\left[{}_aD_t^{-(\gamma-\alpha)}\right] \tag{2}$$

where

$$\Gamma(x) \equiv \int_0^{\infty} y^{x-1}e^{-y}dy \tag{3}$$

is the Gamma function, a and t are limits and α ($\alpha > 0$ and $\alpha \in R$) is the order of the operation. γ is an integer that satisfies $\gamma - 1 < \alpha < \gamma$.

The other approach for fractional order calculus' definition is *Grünwald − Letnikov* definition:

$$_aD_t^{\alpha} = \lim_{\substack{h\to 0\\ nh=t-a}} h^{-\alpha}\sum_{j=0}^{n}(-1)^r\binom{\alpha}{j} f(t-jh) \tag{4}$$

Where the binomial coefficients ($r > 0$)

$$\binom{\alpha}{0} = 1,\ \binom{\alpha}{j} = \frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!} \tag{5}$$

2.2 Laplace and Fourier Transforms

The Laplace transforms [1, 3] of the *Riemann − Liouville* fractional derivative with order $\alpha > 0$ is

$$L\{{}_0D_t^{\alpha}\} = s^{\alpha}F(s) - \sum_{j=0}^{n-1} s^j\left[{}_0D_t^{\alpha-j-1}f(0)\right] \tag{6}$$

where $(n-1) \le \alpha < n$. If

$$_0D_t^{\alpha-j-1}f(0) = 0, \quad j = 0,1,2,\dots,n-1 \tag{7}$$

then

$$L\{{}_0D_t^{\alpha}f(0)\} = s^{\alpha}F(s) \tag{8}$$

Obviously, the Fourier transform of fractional order calculus could be obtained by setting $s = j\omega$ in its Laplace transform just like the classical integer order calculus'.

Fractional order calculus is also a generalization of classical integer order calculus in Laplace and Fourier transforms, which would mean that extremely well developed classical integer order control techniques could still be fully referred in fractional order control.

3 SAMPLING TIME SCALING

By *Riemann − Liouville* definition, fractional order integral with order between 0 and 1 is

$$_0I_t^{\alpha}f(t) = \int_0^t f(\tau)dg_t(\tau),\ 0 < \alpha < 1 \tag{9}$$

where

$$g_t(\tau) = \frac{1}{\Gamma(1+\alpha)}[t^{\alpha} - (t-\tau)^{\alpha}] \tag{10}$$

Let

$$t = nt_s \tag{11}$$

where t_s is the sampling time and n is the step currently under execution. Then,

$$g_{nt_s}(kt_s) = \frac{n^{\alpha} - (n-k)^{\alpha}}{\Gamma(1+\alpha)}t_s^{\alpha},\ k = 1,...,n \tag{12}$$

Therefore, the "real" sampling time T of the kth step in discrete fractional integral controller is

$$T_{nt_s}(kt_s) = \Delta g_{nt_s}(kt_s) = g_{nt_s}(kt_s) - g_{nt_s}[(k-1)t_s] = \frac{(n-k+1)^\alpha - (n-k)^\alpha}{\Gamma(1+\alpha)} {t_s}^\alpha \quad (13)$$

Thus,

$$\begin{aligned} T_{nt_s}(nt_s) &= \frac{1^\alpha - 0^\alpha}{\Gamma(1+\alpha)} {t_s}^\alpha \\ T_{nt_s}[(n-1)t_s] &= \frac{2^\alpha - 1^\alpha}{\Gamma(1+\alpha)} {t_s}^\alpha \\ &\cdots \\ T_{nt_s}(t_s) &= \frac{n^\alpha - (n-1)^\alpha}{\Gamma(1+\alpha)} {t_s}^\alpha \end{aligned} \quad (14)$$

Finally, based on the trapezoidal integration rule

$${}_0I^\alpha_{nt_s} \approx \sum_{k=1}^{n} \frac{f(kt_s) + f[(k-1)t_s]}{2} T_{nt_s}(kt_s) \quad (15)$$

and

$${}_0I^\alpha_{nt_s} = \lim_{\substack{t_s \to 0 \\ t=nt_s}} \sum_{k=1}^{n} \frac{f(kt_s) + f[(k-1)t_s]}{2} T_{nt_s}(kt_s) \quad (16)$$

Similarly, for fractional order derivatives

$${}_0D^\alpha_t f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\alpha} d\tau, \ 0 < \alpha < 1 = \frac{d[\int_0^t f(\tau) dg_t(\tau)]}{dt} \quad (17)$$

where

$$g'_t(\tau) = \frac{1}{\Gamma(2-\alpha)} [t^{1-\alpha} - (t-\tau)^{1-\alpha}] \quad (18)$$

Thus,

$$\begin{aligned} T'_{nt_s}(nt_s) &= \frac{1^{1-\alpha} - 0^{1-\alpha}}{\Gamma(2-\alpha)} {t_s}^{1-\alpha} \\ T'_{nt_s}[(n-1)t_s] &= \frac{2^{1-\alpha} - 1^{1-\alpha}}{\Gamma(2-\alpha)} {t_s}^{1-\alpha} \\ &\cdots \\ T'_{nt_s}(t_s) &= \frac{n^{1-\alpha} - (n-1)^{1-\alpha}}{\Gamma(2-\alpha)} {t_s}^{1-\alpha} \end{aligned} \quad (19)$$

and

$$\int_0^{nt_s} f(\tau) dg'_t(\tau) \approx \sum_{k=1}^{n} \frac{f(kt_s) + f[(k-1)t_s]}{2} T'_{nt_s}(kt_s) \quad (20)$$

Therefore,

$${}_0D^\alpha_{nt_s} f(nt_s) \approx \left\{ \sum_{k=1}^{n} \frac{f(kt_s) + f[(k-1)t_s]}{2} T'_{nt_s}(kt_s) - \sum_{k=1}^{n-1} \frac{f(kt_s) + f[(k-1)t_s]}{2} T'_{(n-1)t_s}(kt_s) \right\} / t_s \quad (21)$$

and

$${}_0D^\alpha_{nt_s} f(nt_s) = \lim_{\substack{t_s \to 0 \\ t=nt_s}} \left\{ \sum_{k=1}^{n} \frac{f(kt_s) + f[(k-1)t_s]}{2} T'_{nt_s}(kt_s) - \sum_{k=1}^{n-1} \frac{f(kt_s) + f[(k-1)t_s]}{2} T'_{(n-1)t_s}(kt_s) \right\} / t_s \quad (22)$$

From Eqn. (14) and Eqn. (19), the interpretation of discrete fractional order controllers is seen to be that they consist in the derivatives and integrals with scaled sampling time decided by the fractional order and the step currently under execution. It can be seen in Fig. 1 and Fig. 2 that the constant sample time is greatly deformed by the step it belongs to. For the discrete $I^{0.5}$ fractional order integral controllers with sampling time 0.01*sec*, the scaled sampling time of the latest step is 0.1128*sec*, over 11 times of the constant sampling time, while in the past 1000th step, it is 0.0018*sec*, only 18 percent of the constant sampling time. The length of scaled sampling time fades away quite rapidly (Fig. 2). Even in past 50th step the scaled sampling time is 0.008*sec*, only 7 percent of the latest one's. The past values are somewhat "forgotten" in discrete fractional order controllers due to the scaled tiny sampling time, while in their integer order counterparts all the values are "remembered" with the same weights.

Clearly, when the order is integer, the existing time scaling effect disappears and all the sampling time is kept constant, rightly equals the classical interpretation of integer order ones, which shows in discrete domain fractional order control is also a generalization and "interpolation" of integer order control theory based on the proposed interpretation.

This time scaling property could explain the robustness of fractional order controllers against saturation and other nonlin-

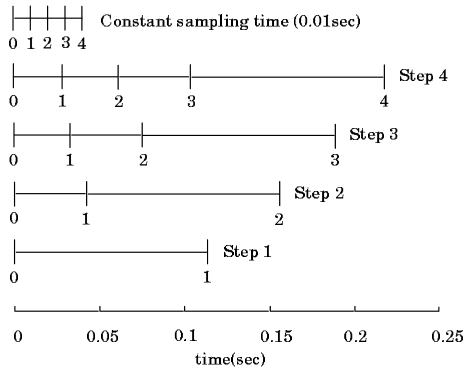

Figure 1. TIME SCALING PROPERTY OF DISCRETE $I^{0.5}$ CONTROLLER

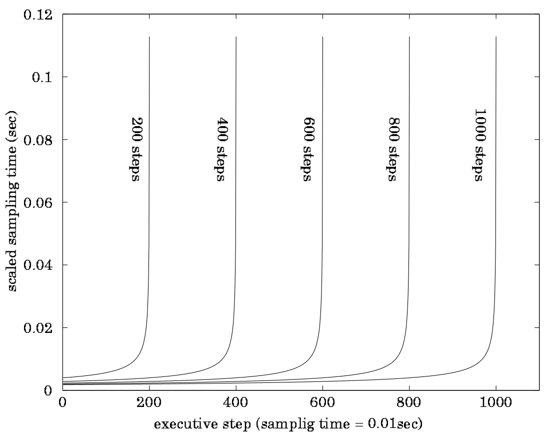

Figure 2. DISCRETE $I^{0.5}$ CONTROLLER'S SCALED SAMPLING TIME

earities and be an easy way to understand the essence of fractional order control systems as being linear time varying system actually. Due to its clear interpretation and easy programming, the property could be a reliable simulation and realization method for some simple fractional order control systems.

The Short Memory Principle is being used intensively in the simulation and realization of discrete fractional order systems by several authors in the literature [3], [10–12]. The principle takes into account the behavior of $f(t)$ only in the "recent past", i.e. in the interval $[t-L, t]$, where L is the length of "memory":

$$ {}_aD_t^\alpha f(t) \approx_{t-L} D_t^\alpha f(t), \; t > (a+L) \tag{23} $$

By using the Short Memory Principle, the discrete equivalent of the fractional order controller in discrete domain is given by

$$ (\omega(z^{-1}))^{\pm\alpha} = T^{\mp\alpha} \sum_{j=0}^{[L/T]} c_j^{(\pm\alpha)} \tag{24} $$

where T is sampling time and the binomial coefficients are:

$$ c_j^{(\pm\alpha)} = (-1)^j \binom{\pm\alpha}{j} = \left(1 - \frac{1+(\pm\alpha)}{j}\right) c_{j-1}^{\pm\alpha}, \; c_0^{\pm\alpha} = 1 \tag{25} $$

However from the sampling time scaling property, it can be clearly seen that even though the past values are forgotten rapidly, due to the large quantities, their influences should not be simply neglected.

In order to show Short Memory Principle's accuracy problem, the fractional α order integral $c(t)$ of the unit step function: $f(t) = 0, \; -\infty < t < 0$; and $f(t) = 1, \; t \geq 0$ simulated by the principle (L=50) is compared with the real values that could be accurately arrived at based on the *Riemann−Liouville* definition:

$$ \begin{aligned} c(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} \cdot 1 \cdot d\xi \\ &= \frac{1}{\Gamma(\alpha)} \cdot \left. \frac{(t-\xi)^\alpha}{\alpha} \right|_t^0 \\ &= \frac{1}{\Gamma(1+\alpha)} t^\alpha \end{aligned} \tag{26} $$

Figure. 3 shows that Short Memory Principle simply discards the past steps' value beyond the memory length and the output becomes saturation immediately. This shortcoming would greatly lower its approximate accuracy and lead to fatal static-state error especially if the fractional order integral controller is realized by this method. At the same time, using the proposed sampling time property could give satisfactory accuracy and be a reliable and easy programming simulation and realization method for discrete fractional order controllers and systems.

4 REALIZATION OF $1/s^\alpha$ SYSTEMS

4.1 Robustness against Gain Variation

The characteristic equation of close-loop $1/s^\alpha$ system with variable gain factor A is

$$ 1 + As^\alpha = 0 \tag{27} $$

For $1 < \alpha < 2$, Eqn. (27) has two complex-conjugate dominant poles in the principle sheet of the *Riemann* surface $-\pi <$

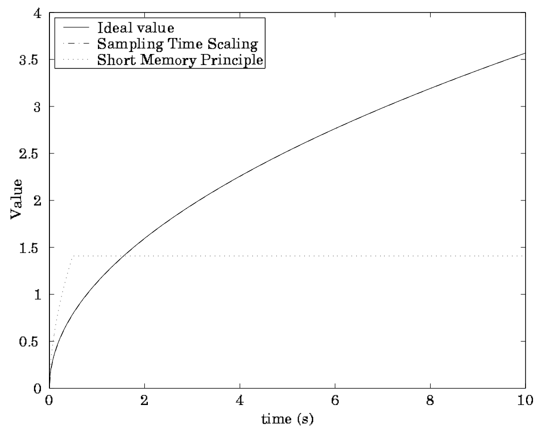

Figure 3. ACCURACY OF SHORT MEMORY PRINCIPLE AND THE PROPOSED METHOD (α=0.5, sampling time=0.01*sec*)

$arg(s) < \pi$:

$$s_{1,2} = A^{\frac{1}{\alpha}} e^{\pm j\pi/\alpha} \tag{28}$$

The relative damping ratio ζ is

$$\zeta = cos\left(\pi - \frac{\pi}{\alpha}\right) = -cos\left(\frac{\pi}{\alpha}\right) \tag{29}$$

This result shows that the relative damping ratio ζ is exclusively decided by order α and independent of the gain factor A.

In frequency domain, the characteristic equation is:

$$1 + AG(j\omega) = 0 \tag{30}$$

Equation. (30) can be rewritten in the form:

$$G(j\omega) = -\frac{1}{A} \tag{31}$$

The movement of $-1/A$ can be considered to be the locus of the critical point (Fig. 4) when the gain variation occurs. For the integer order systems, this movement usually leads to less phase margin and low damping of over-swings. But for fractional $1/s^{\alpha}$ systems, phase margin and relative damping ratio can be kept constant in wide range of frequencies below and in the neighborhood of the critical point. This characteristic highlights the hopeful aspect of applying fraction order controllers to real control problems.

4.2 Time responses of $1/s^{\alpha}$ Systems

In order to reliably verify the robustness of fractional $1/s^{\alpha}$ systems based on the above theoretical analysis, fractional order integral controllers realized by the sampling time scaling property on a digital computer are introduced to achieve the speed control of an electric motor with nominal inertia J_{m0}=6.53×10^{-4}kgm^2, friction coefficient D_m=1.25×10^{-3}$Nm \cdot sec/rad$, controller's coefficient K_i=0.11, sampling time T=0.001sec and an encoder (8000$pulse/rev$) as feedback velocity sensor. Non-linearity of torque saturation is also introduced in the unity feedback control system (Fig. 5).

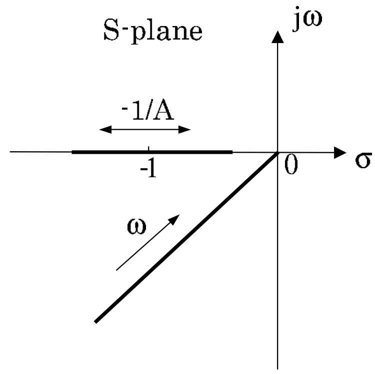

Figure 4. CONSTANT PHASE MARGIN OF $1/s^{\alpha}$ SYSTEM

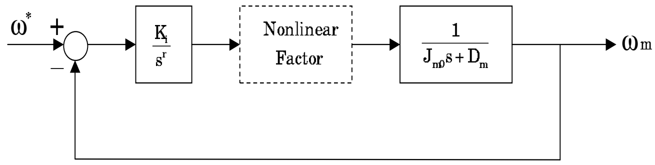

Figure 5. FRACTIONAL *r* ORDER I CONTROL LOOP WITH NON-LINEAR FACTOR

It can be seen clearly that in their time responses, fractional order systems are also the "interpolation" between integer order ones and the time domain performances, such as overshoot and settling time, are changed greatly with different orders (Fig. 6). In Fig. 7 a constant overshoot can be ensured in face of inertia variation, showing a good robustness of fractional $1/s^{\alpha}$ systems in time domain. In the same line, Fig. 8 also shows that fractional order controller is robust for saturation non-linearity, which is one of the most ordinary non-linear phenomena in control systems.

5 CONCLUSIONS

The recent progress in the area of more adequate modeling of control plants by fractional order models reveals promising aspects for future development and application of fractional order control. Some preliminary works have been proposed but

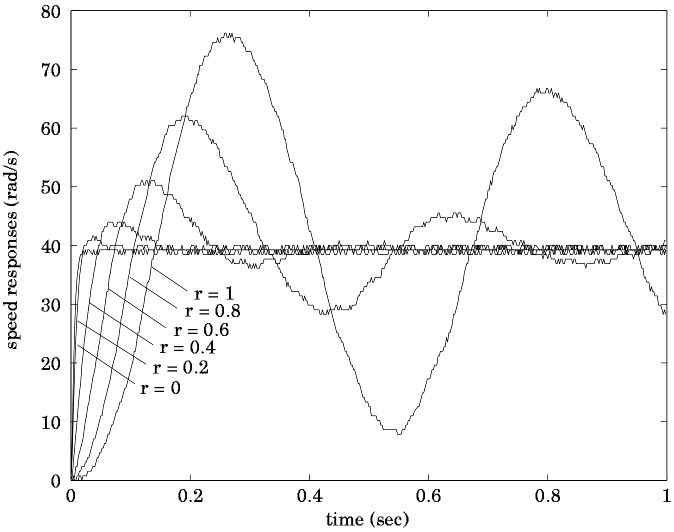

Figure 6. TIME RESPONSES OF $1/s^{\alpha}$ SYSTEM

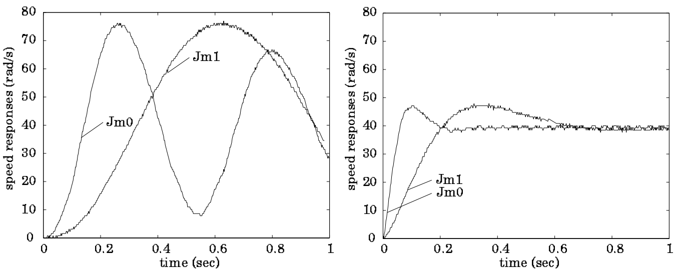

Figure 7. ROBUSTNESS OF THE FRACTIONAL SYSTEM AGAINST INERTIA VARIATION: $J_{m0} = 6.53 \times 10^{-4} kgm^2, J_{m1} = 3.36 \times 10^{-3} kgm^2$ (left: $r = 1$, right: $r = 0.5$)

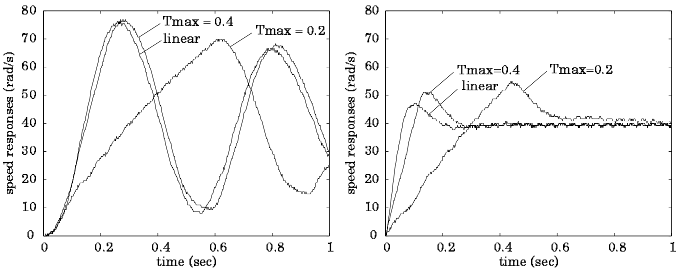

Figure 8. ROBUSTNESS OF THE FRACTIONAL SYSTEM AGAINST TORQUE LIMITATION T_{max} (left: $r = 1$, right: $r = 0.5$)

the clear interpretation of discrete fractional order controllers remained unknown. In this paper, the sampling time scaling property is proposed to give an interpretation of fractional order controllers in discrete domain. This interesting property might be an important point to understand the essence of fractional order control in discrete domain as linear time varying systems and be a helpful hint for exploring the connections between fractional order control and existing modern digital control methods such as multi-rate sampling control and its applications to digital control systems. Some more preferable approximate realization methods of fractional order controllers could also be developed based on this property. Experiments of fractional $1/s^{\alpha}$ systems were also carried out to verify fractional order controllers' superior robustness against parameter variation and saturation non-linearity that highlights their promising aspects while future exploration of the applications to more complex cases is needed.

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