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The Time-Scaled Trapezoidal Integration Rule for Discrete Fractional Order Controllers

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Abstract. In this paper, the time-scaled trapezoidal integration rule for discretizing fractional order controllers is discussed. This interesting proposal is used to interpret discrete fractional order control (FOC) systems as control with scaled sampling time. Based on this time-scaled version of trapezoidal integration rule, discrete FOC can be regarded as some kind of control strategy, in which strong control action is applied to the latest sampled inputs by using scaled sampling time. Namely, there are two time scalers for FOC systems: a normal time scale for ordinary feedback and a scaled one for fractional order controllers. A new realization method is also proposed for discrete fractional order controllers, which is based on the time-scaled trapezoidal integration rule. Finally, a one mass position $1/s^k$ control system, realized by the proposed method, is introduced to verify discrete FOC systems and their robustness against saturation non-linearity.

Key words: discrete, fractional order control, scaled sampling time, trapezoidal integration rule

1. Introduction

The concept of fractional order control (FOC), in which the controlled systems and/or controllers are described by fractional order differential equations, is by no means new. In fact, it has a long history. The concept was firstly introduced by Tustin for the position control of massive objects half a century ago, where the actuator saturation requires sufficient phase margin around and below the crossover frequency [1]. Some other pioneering works were also carried out around 60s [2]. However, FOC was not widely incorporated into control engineering mainly due to the conceptually difficult idea of taking fractional order, the existence of so few physical applications and limited computational power available at that time [3].

In the last few decades, researchers pointed out that fractional order differential equations could model various materials more adequately than integer order ones and provide an excellent tool for describing dynamic processes [4–6]. The fractional order models need fractional order controllers for more effective control of the dynamic systems [7]. This necessity motivated renewed interest in various applications of FOC [8, 9, 10]. Thanks to the rapid development of computational power, modeling and realization of FOC systems also became much easier than before. By changing FOC controller's fractional order, control system's frequency response can be adjusted directly and continuously. This advantage can lead to more straightforward design of robust control systems against uncertainties.

While it is not difficult to understand FOC's theoretical superiority in frequency-domain, control system's performance is more directly measured by its time-domain characteristics. At the same time, it is well known that the discrete integer order controllers have clear time-domain interpretation as the changing ratio or area of sampled input to time, which significantly simplify their use in various applications including control engineering. Classical control theory was extremely well developed based on integer order differential equations. On the contrary, for fractional order controllers, it was not so. Podlubny proposed a simple geometric interpretation of fractional integrals as "changing shadows on the wall" and some pictures describing this changing process were given [11]. However, since most modern controllers are realized by digital computers, clear interpretation of fractional order controllers' roles in discrete domain is much more concerned and with practical importance. Especially insights in discrete fractional order controllers would be enlightening for the future development of FOC researches.

The sampling time scaling property of discrete fractional order controllers was firstly proposed by the authors [12]. In this paper, this interesting property, the time-scaled version of the well known trapezoidal integration rule, is discussed further to gain more insight into discrete FOC systems. Explanation of discrete FOC system as "passive adaptive" control system is proposed. A new realization method, time-scaled trapezoidal integration rule, and its frequency performances with different memory lengths are also investigated. Finally, a one mass position $1/s^k$ control system is introduced to verify discrete FOC systems realized by the time-scaled trapezoidal integral rule. The $1/s^k$ control system's robustness again saturation non-linearity is also examined.

2. Mathematical Aspects

2.1. Mathematical Definitions

The mathematical definition of fractional derivatives and integrals has been the subject of several different approaches [4, 5]. The most frequently encountered definition is called Riemann–Liouville definition, in which the fractional order integrals are defined as

$$ {}_{t_0}D_t^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\xi)^{\alpha-1} f(\xi) d(\xi) \tag{1} $$

while the definition of fractional order derivatives is

$$ {}_{t_0}D_t^{\alpha} = \frac{d^n}{dt^n} \left[{}_{t_0}D_t^{-(n-\alpha)} \right] \tag{2} $$

where

$$ \Gamma(x) \equiv \int_0^{\infty} y^{x-1} e^{-y} dy \tag{3} $$

is the Gamma function, t_0 and t are limits and α ($\alpha > 0$ and $\alpha \in R$) is the order of the operation. n is an integer that satisfies $(n-1) < \alpha < n$.

The other approach is Grünwald–Letnikov definition:

$$ {}_{t_0}D_t^{\alpha} = \lim_{\substack{h \to 0 \\ nh = t - t_0}} h^{-\alpha} \sum_{r=0}^{n} (-1)^{\alpha} \binom{\alpha}{r} f(t - rh) \tag{4} $$

where the binomial coefficients ($r > 0$)

$$ \binom{\alpha}{0} = 1, \qquad \binom{\alpha}{r} = \frac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!} \tag{5} $$

2.2. Laplace and Fourier Transforms

The Laplace transform of Riemann–Liouville fractional order derivative with order $\alpha > 0$ [4, 5] is

$$L\left\{{}_0D_t^\alpha\right\} = s^\alpha F(s) - \sum_{j=0}^{n-1} s^j\left[{}_0D_t^{\alpha-j-1} f(0)\right] \tag{6}$$

where $(n-1) \le \alpha < n$. If

$${}_0D_t^{\alpha-j-1} f(0) = 0, \quad j = 0, 1, 2, \ldots, n-1 \tag{7}$$

then

$$L\left\{{}_0D_t^\alpha f(0)\right\} = s^\alpha F(s) \tag{8}$$

Namely, the Laplace transform of fractional order derivative is fractional order Laplace operator s. Obviously, the Fourier transform of fractional derivative can be obtained by substituting s with $j\omega$ in its Laplace transform just like classical integer order derivatives.

3. Scaled Sampling Time

There are many ways to discretize fractional order controllers for digital implementation. Frequency-band fractional order controller can be realized by broken-line approximation in frequency-domain, but further discretization is required for this method [13]. As to direct discretization, several methods have been proposed such as Short Memory Principle [5], Tustin Taylor expansion [14] and Lagrange function Interpolation method [10], while all these approximation methods need truncation of the expansion series.

However, all the above direct discretization methods for fractional order controllers have a common shortcoming of lacking clear time-domain interpretation. A clear time-domain interpretation was proposed by the authors using sampling time scaling property [12]. Based on this interpretation, discrete fractional order integral can be considered as integral with scaled sampling time; while discrete fractional order derivative is the derivative of the sampling time scaled discrete integral. For completeness, the derivations are repeated below.

From Riemann–Liouville definition, fractional order integral with order between 0 and 1 is

$${}_0I_t^\alpha f(t) = \int_0^t f(\tau)dg_t(\tau), \quad 0 < \alpha < 1 \tag{9}$$

where

$$g_t(\tau) = \frac{1}{\Gamma(1+\alpha)}[t^\alpha - (t-\tau)^\alpha] \tag{10}$$

Let $t := nT$, where T is sampling time and n is the step currently under execution, then

$$g_{nT}\{kT\} = \frac{n^\alpha - (n-k)^\alpha}{\Gamma(1+\alpha)}T^\alpha, \quad k = 1, \ldots, n \tag{11}$$

Based on the same consideration of trapezoidal integration method, the constant sampling time T is adjusted to $T_n(k)$ for the kth step in fractional order discrete integral controller:

$$\begin{aligned}T_n(k) &= \Delta g_{nT}(kT)\\ &= g_{nT}(kT) - g_{nT}[(k-1)T]\\ &= \frac{(n-k+1)^\alpha - (n-k)^\alpha}{\Gamma(1+\alpha)}T^\alpha\end{aligned} \tag{12}$$

Thus

$$\begin{aligned}T_n(n) &= \frac{1^\alpha - 0^\alpha}{\Gamma(1+\alpha)}T^\alpha\\ T_n(n-1) &= \frac{2^\alpha - 1^\alpha}{\Gamma(1+\alpha)}T^\alpha\\ &\cdots\\ T_n(1) &= \frac{n^\alpha - (n-1)^\alpha}{\Gamma(1+\alpha)}T^\alpha\end{aligned} \tag{13}$$

Finally, based on the trapezoidal integration rule

$${}_0I^\alpha_{nT} \approx \sum_{k=1}^{n} \frac{f(kT) + f[(k-1)T]}{2} T_n(k) \tag{14}$$

and if $T \to 0$, then

$${}_0I^\alpha_{nT} = \sum_{k=1}^{n} \frac{f(kT) + f[(k-1)T]}{2} T_n(k) \tag{15}$$

From Equation (14), the interpretation of discrete fractional order integrals is the “deformation” of their integer order counterparts by internal sampling time scaling. As depicted in Figure 1, withthe same sampled inputs $f(kT)$ as integer order integral, the scaled sampling time $T_n(k)$ leads to different value of fractional order integral. Based on this time-scaled version of trapezoidal integration rule, it is easy to understand that the past values are “forgotten” gradually in discrete fractional order integral due to their scaled tiny sampling time, while in integer order ones all the values are “remembered” with same weights.

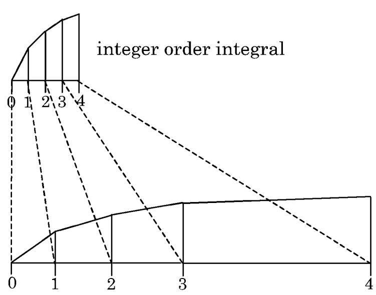

Figure 1. Fractional order integral’s sampling time scaling.

Similarly, discrete fractional order derivatives with order between 0 and 1 is

$$
\begin{aligned}
{}_0D_t^{\alpha} f(t) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha}} \mathrm{d}\tau \\
&= \frac{d[\int_0^t f(\tau) dg_t'(\tau)]}{dt}, \quad 0 < \alpha < 1
\end{aligned} \tag{16}
$$

where

$$
g_t'(\tau) = \frac{1}{\Gamma(2-\alpha)} [t^{1-\alpha} - (t-\tau)^{1-\alpha}] \tag{17}
$$

Thus

$$
\begin{aligned}
T_n'(n) &= \frac{1^{1-\alpha} - 0^{1-\alpha}}{\Gamma(2-\alpha)} T^{1-\alpha} \\
T_n'(n-1) &= \frac{2^{1-\alpha} - 1^{1-\alpha}}{\Gamma(2-\alpha)} T^{1-\alpha} \\
&\cdots \\
T_n'(1) &= \frac{n^{1-\alpha} - (n-1)^{1-\alpha}}{\Gamma(2-\alpha)} T^{1-\alpha}
\end{aligned} \tag{18}
$$

Again based on the trapezoidal integration rule

$$
\int_0^{nT} f(\tau) dg_t'(\tau) \approx \sum_{k=1}^{n} \frac{f(kT) + f[(k-1)T]}{2} T_n'(k) \tag{19}
$$

and if $T \to 0$, then

$$
\int_0^{nT} f(\tau) dg_t'(\tau) = \sum_{k=1}^{n} \frac{f(kT) + f[(k-1)T]}{2} T_n'(k) \tag{20}
$$

The interpretation of discrete fractional order derivatives is the derivatives of fractional $(1-\alpha)$ order integrals $\int_0^{nT} f(\tau) dg_t'(\tau)$. Namely, it can be understood geometrically as the changing ratio of the "scaled integral area" due to the scaled sampling time, as depicted in the shadow area of Figure 2.

Clearly, when the discrete controller's order α equals 1, the sampling time will not be scaled any more. From the viewpoint of sampling time scaling, in discrete domain FOC is also a generalization and "interpolation" of the classical integer order control theory.

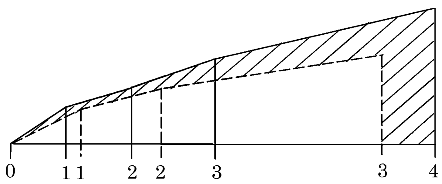

Figure 2. Changing of the "scaled integral area".

4. Control with Scaled Sampling Time

Viewing in terms of sampling time scaling can gain more insight into discrete FOC systems. The fractional order controllers are controllers with self-adjustable parameters and a mechanism for adjusting the parameters. As depicted in Figure 3, a fractional order controller can be considered as the series of sampling time scaler and classical integer order controller conceptually. Namely, the sampling time of input sequence is pre-adjusted by sampling time scaler before entering integer order controller.

Therefore, fractional order control can be regarded as a special control strategy, which apply strong control action to latest sampled inputs by using "forgetting factors" $\lambda_n(k)$. Large scaled sampling time of latest values means small "forgetting factors" and vice versa. For example, the control law of a pure fractional order integral controller can be rewritten in "forgetting factor" form, where $\lambda_n(k)$ equal $2/T_n(k)$ in Equation (14):

$$u(n) = \sum_{k=1}^{n} \frac{1}{\lambda_n(k)}[e(k) + e(k-1)] \tag{21}$$

It can be seen in Figure 4 that in fractional order integral controllers the input values are memorized with time-scaled weights, while the integer order controllers give all the values with same weights. Farther fractional order differs from the integer order 1, more obvious the sampling time is scaled. The rapidly fading influences of the old values and dominance of the latest ones make fractional order controllers "passively adaptive" to present changes of dynamic processes. This can also be a time domain explanation for FOC systems' robustness against uncertainties.

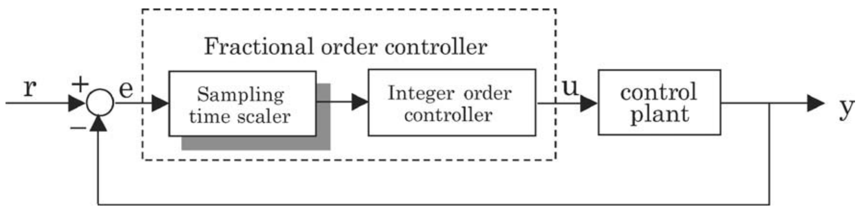

Figure 3. Sampling time scaler of FOC systems.

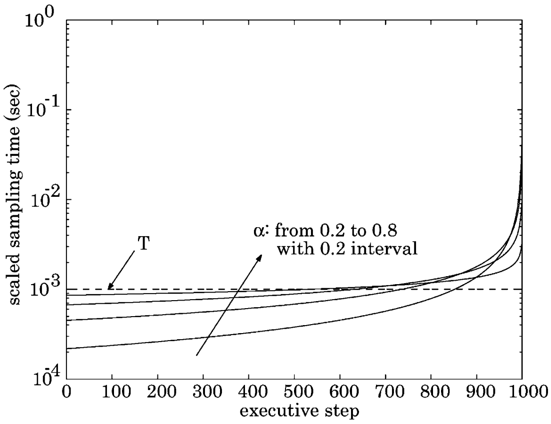

Figure 4. Discrete I^α controller's scaled sampling time ($T = 0.001s$).

5. Realization by Scaled Sampling Time

It is common knowledge that the fractional order systems have an infinite dimension while the integer order systems are finite dimensional. Discretization of fractional order controllers by the time-scaled trapezoidal integration rule is not an exception. Proper approximation by finite difference equation is needed to realize fractional order controllers. Based on the observation that the lengths of scaled sampling time near "starting point" t_0 is small enough to be "forgotten" for large t (see Figure 4), a new realization method is proposed to take into account only the behavior of $f(t)$ in the "recent past", i.e., in the interval $[t-L, t]$, where L is the length of "memory":

$$ {}_{t_0}D_t^k f(t) \approx {}_{t-L} D_t^k f(t), \quad t > t_0 + L \tag{22}$$

Therefore this realization method can be considered as a kind of "short memory principle" approach but based on Riemann–Liouville definition [5].

From Equations (14) and (19), it is easy to give the discrete equivalent of fractional α order integral or derivative controllers as follows:

$$ Z(D^\alpha[x(t)]) \approx \frac{1}{T^\alpha} \sum_{j=0}^{[L/T]} c_j z^{-j} \tag{23}$$

For integral controllers ($\alpha < 0$), coefficients c_j are

$$ \begin{aligned} c_0 &= \frac{1}{2\Gamma(1+|\alpha|)} \\ c_j &= \frac{(j+1)^{|\alpha|} - (j-1)^{|\alpha|}}{2\Gamma(1+|\alpha|)}, \quad j \geq 1 \end{aligned} \tag{24}$$

And the coefficients of derivative controllers ($\alpha > 0$) are

$$ \begin{aligned} c_0 &= \frac{1}{2\Gamma(2-\alpha)} \\ c_1 &= \frac{2^{1-\alpha} - 1}{2\Gamma(2-\alpha)} \\ c_j &= \frac{1}{2\Gamma(2-\alpha)}[(j+1)^{1-\alpha} - j^{1-\alpha} - (j-1)^{1-\alpha} - (j-2)^{1-\alpha}], \quad j \geq 2 \end{aligned} \tag{25}$$

Figure 5 shows the Bode plot of discrete $Z(1/s^{0.5})$ controller for sampling time $T = 0.001s$ realized by different [L/T] (solid) compared with the ideal case of continuous controller $1/s^{0.5}$ (dash). Clearly, in order to have a better approximation in discrete domain, smaller sampling time and larger $[L/T]$ (memory length) are preferable.

6. Example: One Mass Position Control

In order to verify discrete FOC systems realized by proposed realization method, one mass position control is introduced as a simple prototype, where $J_m = 0.001\text{kgm}^2$ and $K_d = 0.01$ (see Figure 6).

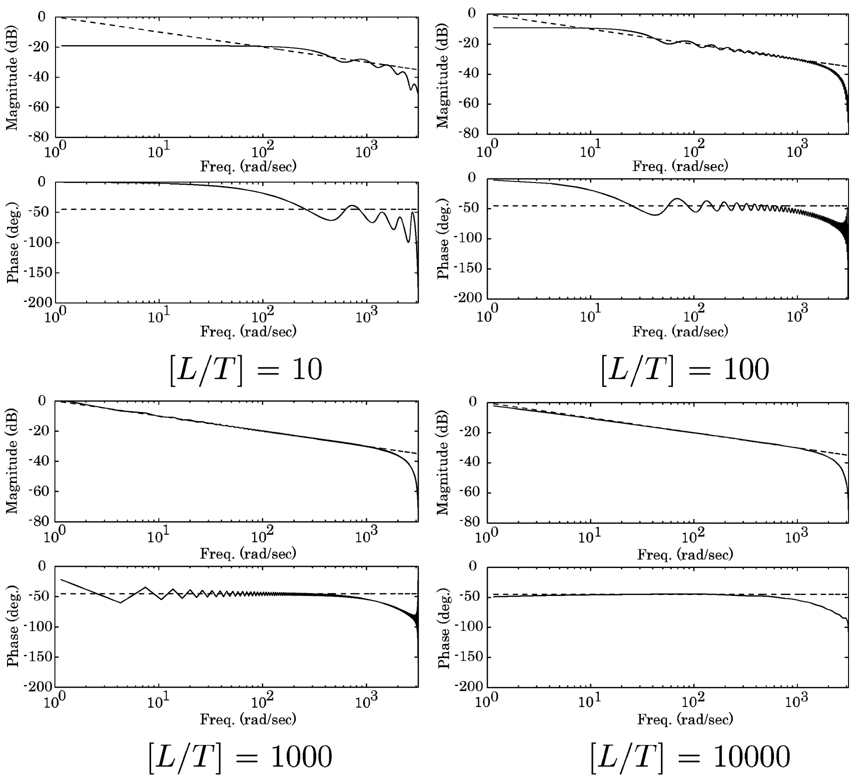

Figure 5. Bode plots of $Z(1/s^{0.5})$ with different memory lengths.

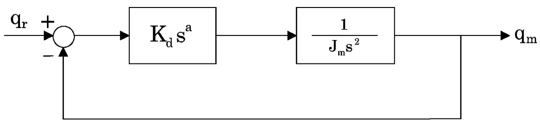

Figure 6. The position control loop with fractional α order derivative controller.

Time responses with fractional order derivative controllers D^α are simulated using full memory length. Namely, all the past sampled input will be remembered.

The time responses with different α order derivative controllers are depicted in Figure 7. It can be seen clearly that the FOC systems' time responses are interpolations of the classical integer order ones. The scaled sampling time directly leads to the change of the time responses. As depicted in the figure, the important quantities of step response, maximum overshoot, delay time, rise time and settling time, can be adjusted continuously by changing fractional order α.

For one mass position control with D^α controllers, its open-loop's phase margin is $180-(2-\alpha)\times 90$ degree. That is the phase margin can be continuously adjustable by changing fractional order α. This superiority of FOC in frequency domain make control systems robust against uncertainties, especially gain variation. The time responses also give same results (see Figure 8). An output torque limitation of $\pm 2NM$ is introduced to the α order derivative controllers. Comparison of the responses shows that fractional order $1/s^k$ ($k = 2 - \alpha$) system's robustness against saturation non-linearity can be adjusted continuously between the classical integer order ones. Among them, $1/s^{1.6}$ system ($\alpha = 0.4$) has the best time-domain performance.

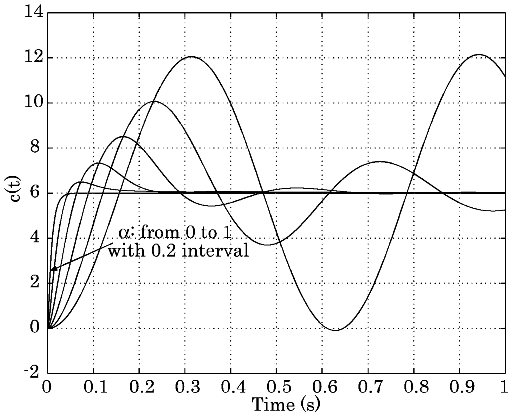

Figure 7. Time responses with fractional α order derivative controllers.

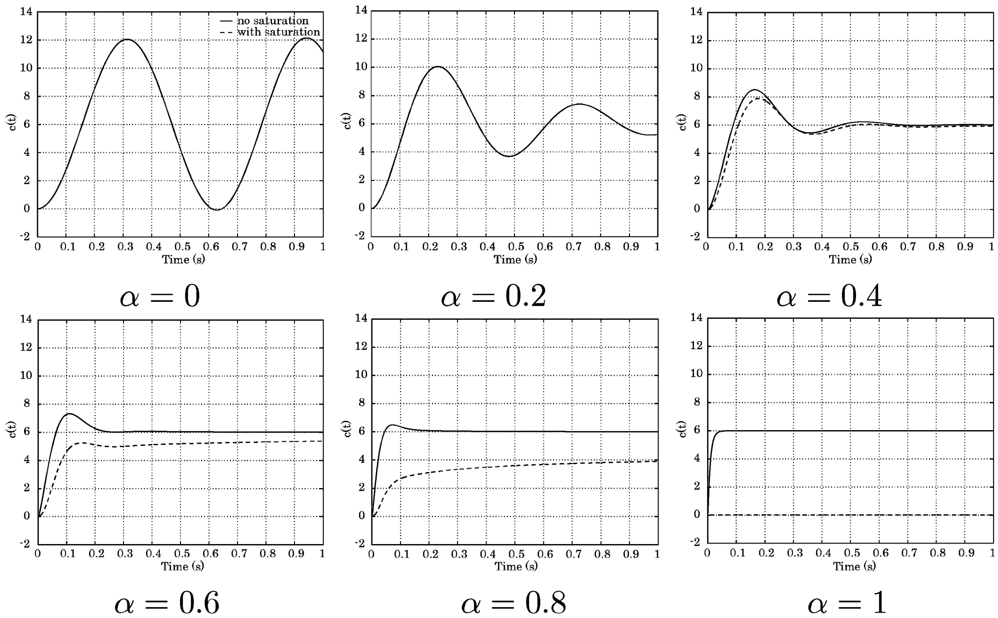

Figure 8. Time responses with saturation non-linearity.

This proposed realization method was also applied to implement fractional order D^{α} controller for gear backlash vibration suppression in torsional system's speed control, where 100 past values are memorized (Ma and Hori, submitted for publication). In real applications, the necessary memory length, namely how good the approximation is needed, should be decided by the demand of specific control problem.

7. Conclusions

In this paper, the time-scaled trapezoidal integration rule for discretizing fractional order controller is discussed. This interesting proposal is used to interpret discrete FOC systems as control with scaled sampling time. A new realization method is also given for discrete fractional order controllers with clear geometric interpretation. Based on this time-scaled version of the well-known trapezoidal integration rule, discrete FOC can be regarded as some kind of control strategy, in which strong control action is applied to the latest sampled inputs by sampling time scaling. Namely, there are two time scalers for FOC systems: a normal time scale for ordinary feedback and a scaled one for fractional order controllers. The explanation of discrete FOC as "passive adaptive control" can give more insight into understanding the essence of FOC and its robustness against uncertainties; while further research is still needed in this field.

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